Combinatorics

What is Combinatorics?

- Wide ranging field involving the study of discrete objects

What is Combinatorics?

- Wide ranging field involving the study of discrete objects
- Enumerative Combinatorics
  - counting the objects that satisfy certain criteria

Permutations and Combinations

- Permutations
  - Number of ways of arranging a list of elements
  - Order is important
- Combinations
  - Number of ways of selecting k elements from a set
  - Order is unimportant

Permutations and Combinations

- Permutations with repetition
  - n^k
- Permutations without repetition
  - $P(n,k) = \frac{n!}{(n-k)!}$
- Combinations without repetition (Binomial Coefficients)
  - $C(n,k) = \binom{n}{k} = \frac{P(n,k)}{k!} = \frac{n!}{k!(n-k)!}$

Pascal's Triangle

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Row 0 1

Row 1 1 1

Row 2 1 2 1

Row 3 1 3 3 1

Row 4 1 4 6 4 1

Row 5 1 5 10 10 5 1

Row 6 1 6 15 20 15 6 1

Row 7 1 7 21 35 35 21 7 1

....

Combinations with repetition

- Combinations with repetition
  - $\left(\!\!\binom{n}{k}\!\!\right) = \binom{n+k-1}{k} = \frac{(n+k-1)!}{k!(n-1)!}$

Multinomial Coefficients

- Determine the coefficients of the expansion of

$$(x_1+x_2+\cdots+x_m)^n = \sum_{k_1,k_2,\ldots,k_m} \binom{n}{k_1,k_2,\ldots,k_m} x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m}$$

- The number of ways of placing n objects into m boxes, the i^{th} one being of size k_i
- The number of permutations of a string with length n and m distinct letters and with k_i denoting the number of times the i^{th} letter appears.
- $$\binom{n}{k_1,k_2,k_3,\ldots,k_m} = \frac{n!}{k_1!k_2!k_3!\cdots k_m!}$$ Note: $\sum_{i=1}^{m} k_i = n$

Stirling numbers

- How many ways are there to partition a set of size n into k non-empty subsets?

$$S(n,k) = S_n^{(k)} = \left\{ \begin{matrix} n \\ k \end{matrix} \right\}$$

- How do you calculate S(n, k)?

Stirling numbers

- Split problem into disjoint sub-problems
- A partition either contains the n^{th} element by itself or as part of another set
- When element is alone the number of other partitions is S(n-1, k-1)
- When elements is part of another set there are S(n-1, k) ways to partition the set without it and k sets to insert it into
- Answer: $S(n, k) = S(n-1, k-1) + k.S(n-1, k)$

Stirling numbers

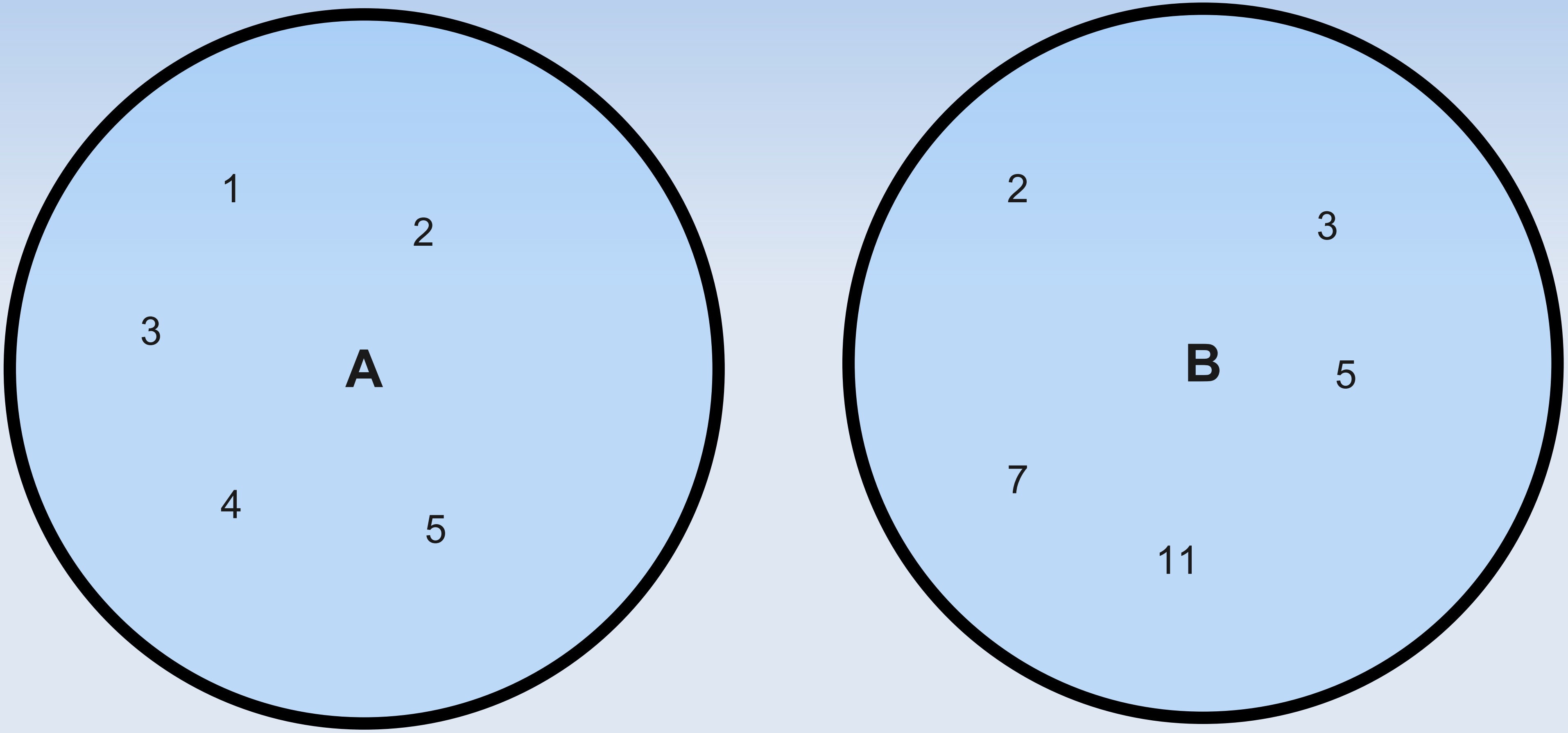
$$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\} \quad \text{where} \quad \left\{ {n \atop 1} \right\} = \left\{ {n \atop n} \right\} = 1$$

Inclusion Exclusion

- If set A has 5 elements and set B has 5 elements how many elements does the union have?

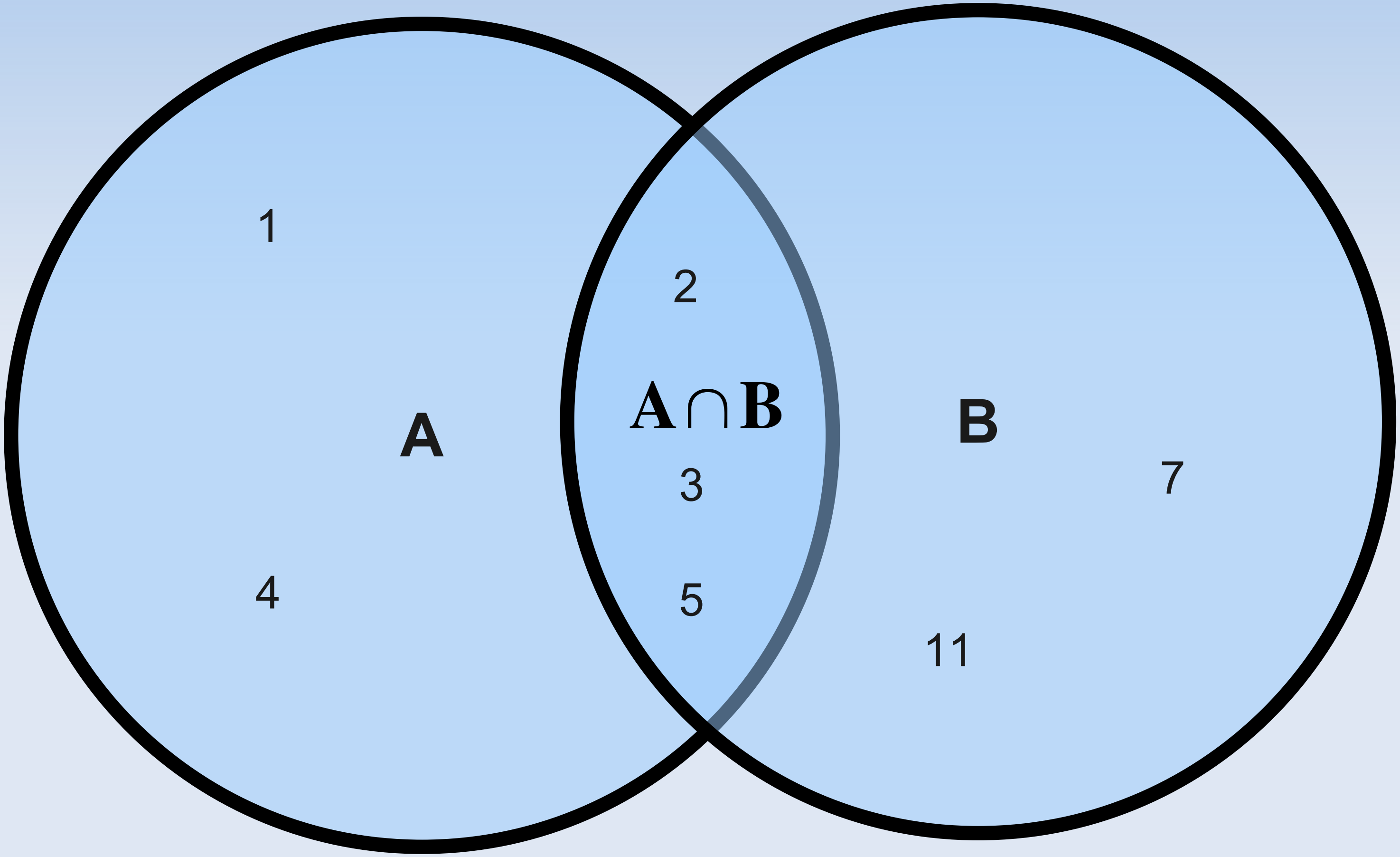
Inclusion Exclusion

Inclusion Exclusion



5+5 = 10

Inclusion Exclusion

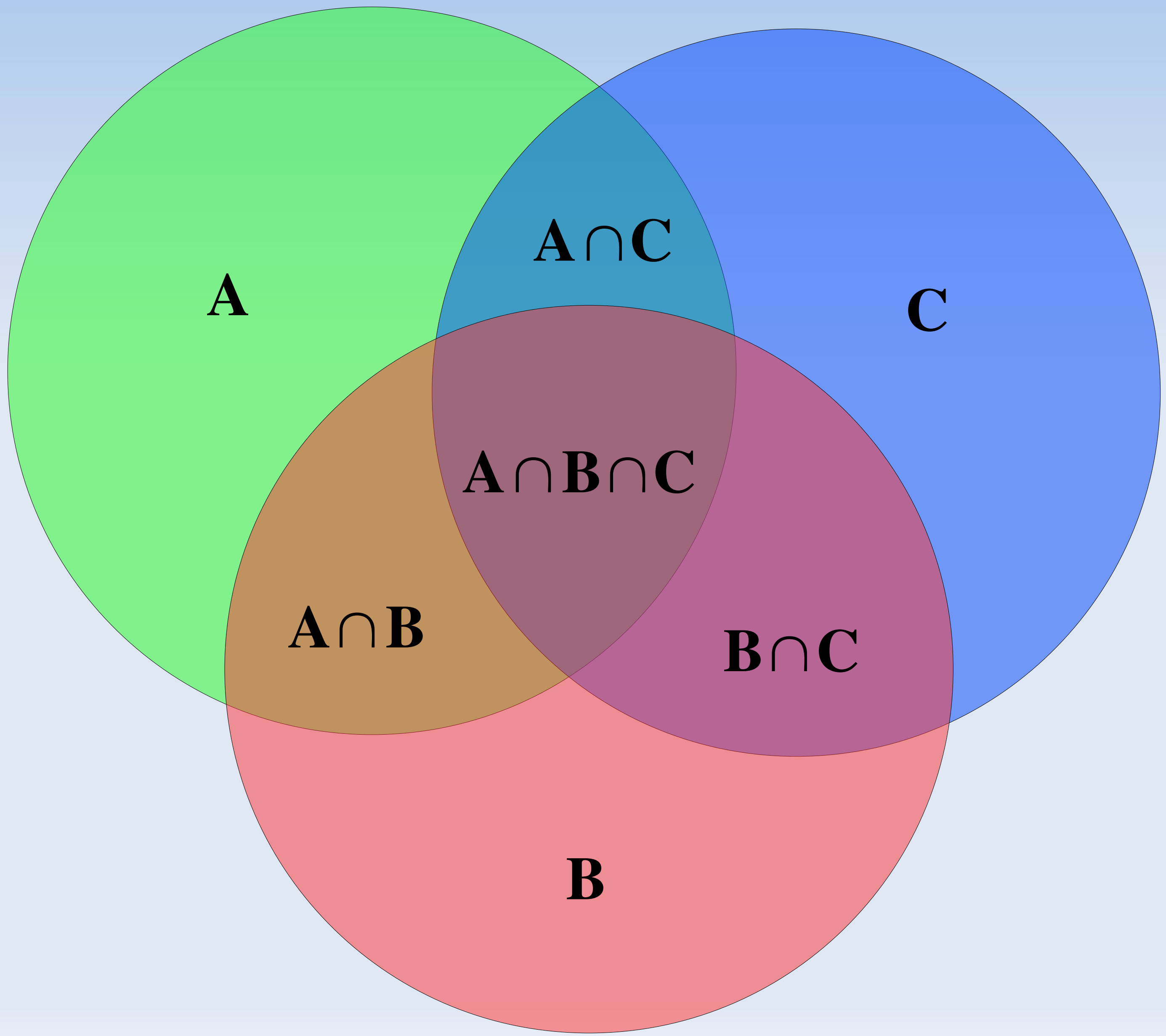


$$\left|\mathbf{A} \cup \mathbf{B}\right| = 7$$

Inclusion Exclusion

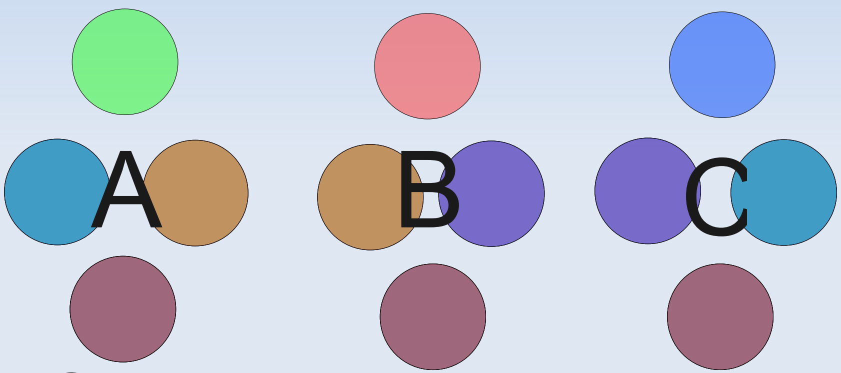
- $\left|\mathbf{A} \cup \mathbf{B}\right| = \left|\mathbf{A}\right| + \left|\mathbf{B}\right| - \left|\mathbf{A} \cap \mathbf{B}\right|$

Inclusion Exclusion

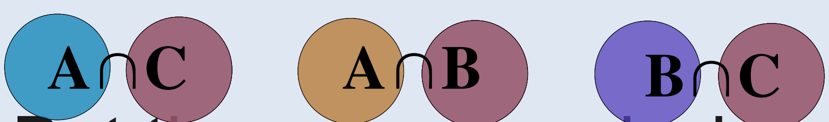


Inclusion Exclusion

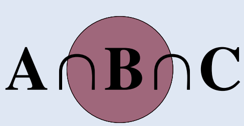
Adding the the sizes of each set will give:

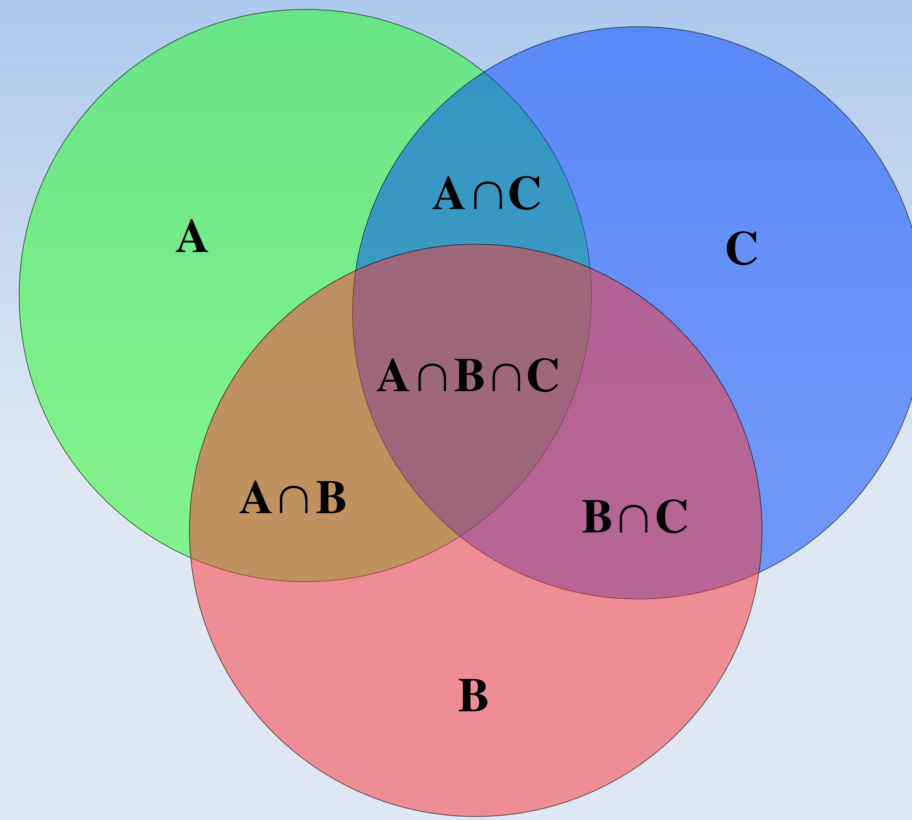


So we need to subtract



But then we are missing one so we add it





Inclusion Exclusion

- $\left|\mathbf{A} \cup \mathbf{B}\right| = \left|\mathbf{A}\right| + \left|\mathbf{B}\right| - \left|\mathbf{A} \cap \mathbf{B}\right|$
- $\left|\mathbf{A} \cup \mathbf{B} \cup \mathbf{C}\right| = \left|\mathbf{A}\right| + \left|\mathbf{B}\right| + \left|\mathbf{C}\right| - \left|\mathbf{A} \cap \mathbf{B}\right| - \left|\mathbf{A} \cap \mathbf{C}\right| - \left|\mathbf{B} \cap \mathbf{C}\right| + \left|\mathbf{A} \cap \mathbf{B} \cap \mathbf{C}\right|$

Inclusion Exclusion

- $|\mathbf{A} \cup \mathbf{B}| = |\mathbf{A}| + |\mathbf{B}| - |\mathbf{A} \cap \mathbf{B}|$
- $|\mathbf{A} \cup \mathbf{B} \cup \mathbf{C}| = |\mathbf{A}| + |\mathbf{B}| + |\mathbf{C}| - |\mathbf{A} \cap \mathbf{B}| - |\mathbf{A} \cap \mathbf{C}| - |\mathbf{B} \cap \mathbf{C}| + |\mathbf{A} \cap \mathbf{B} \cap \mathbf{C}|$

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{i,j\,:\,1 \le i < j \le n} |A_i \cap A_j| + \sum_{i,j,k\,:\,1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n-1} |A_1 \cap \cdots \cap A_n|$$

Inclusion Exclusion

- COCI Contest #4 Question 5 (Simplified)
- There is an $n \times m$ grid with safes at each grid position
- $n <= 2000$; $m <= 1000\,000\,000$
- There's a guard at bottom left-hand corner (0,0)
- How many safes can he see?
  - He can see the ones at (x,y) such that x and y are co-prime

Inclusion Exclusion

For each i = 1 to n

- Factorise i into prime factors $f_1, f_2, \ldots, f_k$
- Let M_j be the set of multiples of f_j less than or equal to m
- Position (i, x), such that $x \in M_j$, can't be seen because i and x necessarily share a factor.
- Let $X = M_1 \cup M_2 \cup \ldots \cup M_k$

  Note: $|M_j| = \left\lfloor \frac{m}{f_j} \right\rfloor$
- Therefore X contains all of the positions that can't be seen
- And |X'| = m-|X| safes can be seen in this column
- Add this to the total.

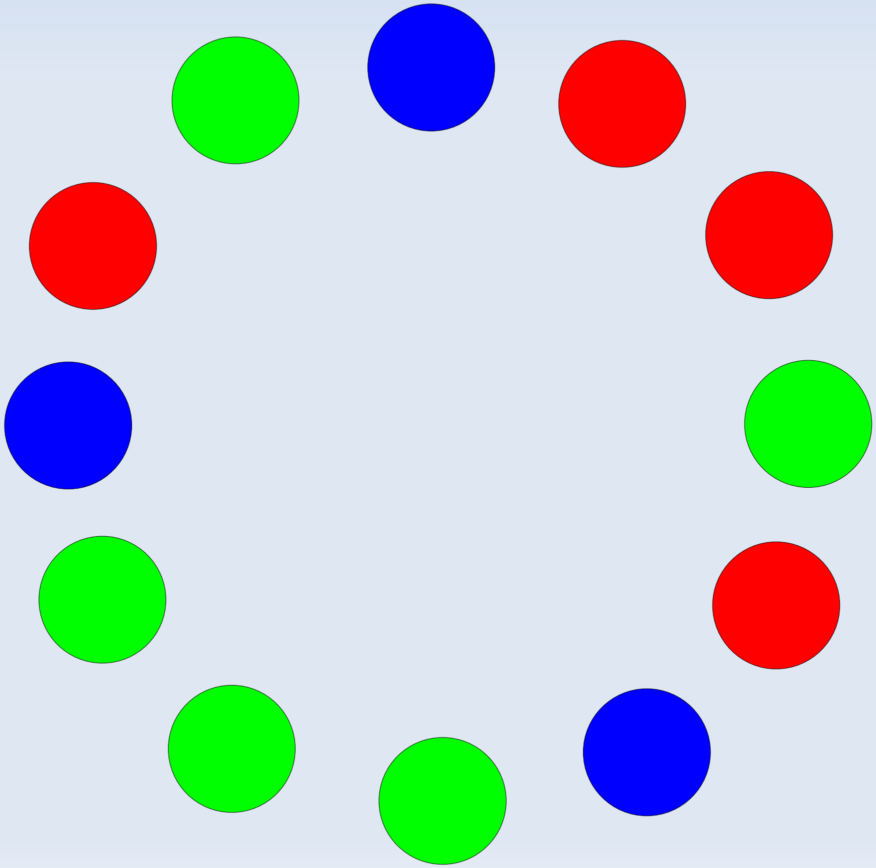
Inclusion Exclusion

- Note: inclusion-exclusion principal is slow:

  $O(2^k)$ where k is the number of sets.
- But in many cases it works.
- In this case, there are at most 4 distinct prime factors, so it easily runs in time.
  - The maximum number of distinct factors is 4, because the smallest number with 5 distinct prime factors is the product of the first 5 primes: 2*3*5*7*11 = 2310 and n can only go up to 2000.

Burnside's lemma

- How many ways are there to colour a ring of 12 objects with 3 colours taking rotations into account?



Burnside's lemma

- How many ways are there to colour a ring of 12 objects with 3 colours taking rotations into account?
- If you ignore rotations it is trivial: 3^{12}

Burnside's lemma

- How do you take equivalent solutions into account?

Burnside's lemma

- Decide what makes solutions equivalent.
  - These are the transformation that you can perform on a solution without changing it into another distinct solution.

Burnside's lemma

- Decide what makes solutions equivalent.
  - These are the transformation that you can perform on a solution without changing it into another distinct solution.

- In this case they are
  - *Identity* - NB
  - Rotate right 1
  - Rotate right 2
  - Rotate right 3
  - Rotate right 4
  - Rotate right 5
  - Rotate right 6
  - Rotate right 7
  - Rotate right 8
  - Rotate right 9
  - Rotate right 10
  - Rotate right 11

Burnside's lemma

- Consider each transform
  - How many solutions remain unchanged after the transform is applied?
- Sum the results for each transform and divide by the number of transforms.

Burnside's lemma

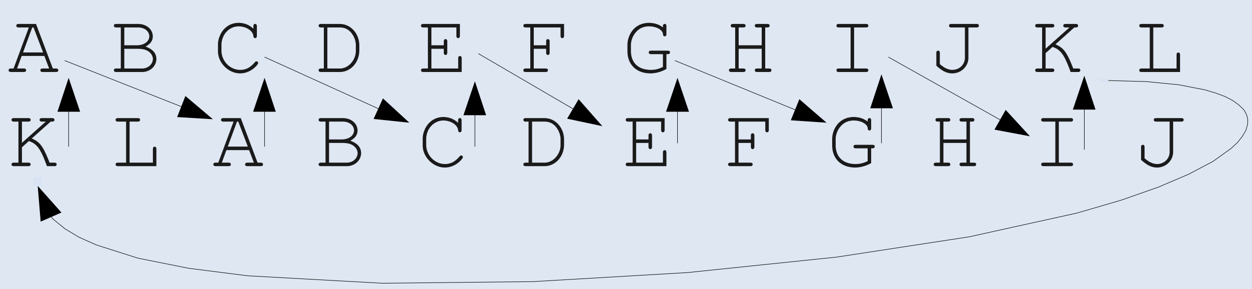
- Identity
  - All solutions remain the same (by definition): 3^{12}
- Rotate right 1

```
A B C D E F G H I J K L
L A B C D E F G H I J K
```

For the solution not to change: A must equal L, B must equal A, etc. Because A = L and B = A, B = L, etc. Therefore they are all equal.

Burnside's lemma

- Identity
  - All solutions remain the same (by definition): ٣^{12}
- Rotate right 1
  - Only the solutions with all objects equal remain the same: 3^1
- Rotate right 2



Burnside's lemma

- Identity
  - All solutions remain the same (by definition): $3^{١٢}$
- Rotate right 1
  - Only the solutions with all objects equal remain the same: 3^{1}
- Rotate right 2
  - Only the solutions with the same objects in 2 cycles remain the same: ٣^{2}

Burnside's lemma

- Rotate right 3
  - Only the solutions with the same objects in 3 cycles remain the same: 3^3
- Rotate right 4
  - Only the solutions with the same objects in 4 cycles remain the same: ٣^4
- Rotate right 5
  - Only the solutions with all objects equal remain the same:

$$3^1$$

Burnside's lemma

- Rotate right 6
  - Only the solutions with the same objects on opposite sides of ring: 3^6
- Rotate right 7 = Rotate left 5 = Rotate right 5
- Rotate right 8 = Rotate left 4 = Rotate right 4
- Rotate right 9 = Rotate left 3 = Rotate right 3
- Rotate right 10 = Rotate left 2 = Rotate right 2
- Rotate right 11 = Rotate left 1 = Rotate right 1

Burnside's lemma

- Sum up the results and divide
  - $$\frac{3^{12}+\text{٤}.\text{٣}^{1}+2.3^{2}+2.3^{3}+2.3^{4}+3^{6}}{12}=44368$$

Burnside's lemma

- Sum up the results and divide
  - $$\frac{3^{12}+4.3^{1}+2.3^{2}+2.3^{3}+2.3^{4}+3^{6}}{12}=44368$$

It's that easy: just bear in mind the numbers you sum are not always going to be powers. In fact, you may need to think about each transform separately.

Calculating Combinations

```
function combination( n, k )
   c = 1
   for i = 0 to k-1
      c = c * (n-i) / (i+1) // not *=
   return c
```

Calculating Combinations

- Another method is to pre-compute Pascal's triangle and use it as a look-up table for combinations.
- This is useful if many different combinations need to be computed, as long as n does not get too large.

TRICKS

- Dividing by k! to remove permutations
- Splitting problem into 2 disjoint sub-problems
- Distinguished Element – make one element special
- Inclusion-exclusion
- Burnside's lemma
- DP